Recall from lecture:

$$\underline{f}^{Roe}_{i+1/2} = \frac{1}{2}\left(\underline{f}_i + \underline{f}_{i+1}\right) - \frac{1}{2}\,|\underline{\underline{A}}|\left(\underline{Q}_{i+1} - \underline{Q}_i\right) \qquad (*)$$

$$\underline{Q}_i^{n+1} = \underline{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\underline{f}^{Roe}_{i+1/2} - \underline{f}^{Roe}_{i-1/2}\right)$$

Why can't we apply this yet for non-conservative systems?

→ We are given

$$\frac{\partial \underline{Q}}{\partial t} + \underline{\underline{A}}(\underline{Q})\,\frac{\partial \underline{Q}}{\partial x} = 0$$

<u>without</u> the ability to transform

$$\underline{\underline{A}}(\underline{Q})\,\frac{\partial \underline{Q}}{\partial x} = \frac{\partial \underline{f}}{\partial \underline{Q}}\,\frac{\partial \underline{Q}}{\partial x} = \frac{\partial \underline{f}}{\partial x}$$

into $\underline{\underline{A}}(\underline{Q}) = \underline{\underline{\frac{\partial f}{\partial Q}}} + \underline{\underline{NC}}$

How to proceed:

Add a "0": $0 = +\underline{f}_i - \underline{f}_i$

$$\Rightarrow \underline{Q}_i^{n+1} = \underline{Q}_i^{n} - \frac{\Delta t}{\Delta x}\Big(\underbrace{\underline{f}_{i+1/2}^{Roe} - \underline{f}_i}_{(A)} + \underbrace{\underline{f}_i - \underline{f}_{i-1/2}^{Roe}}_{(B)} \Big)$$

(A) Plug in (*)

$$(A) = \left(\frac{1}{2}\left(\underline{f}_{i+1} + \underline{f}_i\right) - \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_{i+1} - \underline{Q}_i\right) \right) - \underline{f}_i$$

$$= \frac{1}{2}\left(\underline{f}_{i+1} - \underline{f}_i\right) - \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_{i+1} - \underline{Q}_i\right)$$

$$= \frac{1}{2}\underline{\underline{A}}\left(\underline{Q}_{i+1} - \underline{Q}_i\right) - \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_{i+1} - \underline{Q}_i\right)$$

$$(B) = \underline{f}_i - \left(\frac{1}{2}\left(\underline{f}_i + \underline{f}_{i-1}\right) - \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_i - \underline{Q}_{i-1}\right) \right)$$

$$= \frac{1}{2}\left(\underline{f}_i - \underline{f}_{i-1}\right) + \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_i - \underline{Q}_{i-1}\right)$$

$$= \frac{1}{2}\underline{\underline{A}}\left(\underline{Q}_i - \underline{Q}_{i-1}\right) + \frac{1}{2}\left|\underline{\underline{A}}\right|\left(\underline{Q}_i - \underline{Q}_{i-1}\right)$$

$$\Rightarrow \underline{Q}_i^{n+1} = \underline{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\underbrace{\underline{f}^{Roe}_{i+1/2} - \underline{f}_i}_{= \underline{D}^-_{i+1/2}} + \underbrace{\underline{f}_i - \underline{f}^{Roe}_{i-1/2}}_{= \underline{D}^+_{i-1/2}} \right)$$

" Fluctuations "

Remarks: Note the sign $\longrightarrow +$, which is positive !

$$\underline{D}^-_{i+1/2} := (A) = \frac{1}{2}\underline{\underline{A}}(\underline{Q}_{i+1} - \underline{Q}_i) - \frac{1}{2}|\underline{\underline{A}}|(\underline{Q}_{i+1} - \underline{Q}_i)$$

$$\underline{D}^+_{i-1/2} := (B) = \frac{1}{2}\underline{\underline{A}}(\underline{Q}_i - \underline{Q}_{i-1}) + \frac{1}{2}|\underline{\underline{A}}|(\underline{Q}_i - \underline{Q}_{i-1})$$

Code:
- We require $\underline{\underline{A}} = \frac{\partial f}{\partial \underline{\underline{Q}}} + \underline{\underline{NC}}$
- We require $|\underline{\underline{A}}|$.

if $\underline{\underline{A}} = \underline{\underline{R}}\,\underline{\underline{\Lambda}}\,\underline{\underline{R}}^{-1}$, then

$$|\underline{\underline{A}}| = \underline{\underline{R}}\,|\underline{\underline{\Lambda}}|\,\underline{\underline{R}}^{-1}$$

2D - quadrilateral FVM?

$$\underline{Q}_i^{n+1} = \underline{Q}_i^n - \left(\underline{g}_{Ni} - \underline{g}_{Si} + \underline{f}_{ei} - \underline{f}_{wi} \right)$$

$$\mp \underline{g}_i \qquad\qquad -+ \underline{f}_i$$

For the PDE:

$$\frac{\partial \underline{Q}}{\partial t} + \underline{\underline{A}}(\underline{Q}) \frac{\partial \underline{Q}}{\partial x} + \underline{\underline{B}}(\underline{Q}) \frac{\partial \underline{Q}}{\partial y} = 0$$

it follows that

$$\underline{g}_{Ni} - \underline{g}_i = \underline{D}_N^- = \frac{1}{2} \left(\underline{\underline{B}} (\underline{Q}_N - \underline{Q}_i) - |\underline{\underline{B}}| (\underline{Q}_N - \underline{Q}_i) \right)$$

$$\underline{g}_i - \underline{g}_{Si} = \underline{D}_S^+ = \frac{1}{2} \left(\underline{\underline{B}} (\underline{Q}_i - \underline{Q}_S) + |\underline{\underline{B}}| (\underline{Q}_i - \underline{Q}_S) \right)$$

$$\underline{f_{ei}} - \underline{f_i} = \underline{D_e^-} = \frac{1}{2}\left(\underline{\underline{A}}(\underline{Q_e} - \underline{Q_i}) - |\underline{\underline{A}}|(\underline{Q_e} - \underline{Q_i})\right)$$

$$\underline{f_i} - \underline{f_{wi}} = \underline{D_w^+} = \frac{1}{2}\left(\underline{\underline{A}}(\underline{Q_i} - \underline{Q_w}) + |\underline{\underline{A}}|(\underline{Q_i} - \underline{Q_w})\right)$$

Recall:

The matrices $\underline{\underline{A}} = \underline{\underline{A}}(\underline{Q_i}, \underline{Q_j})$

$\underline{\underline{B}} = \underline{\underline{B}}(\underline{Q_i}, \underline{Q_j})$

are defined via the path-integral.

E.g.

$$\underline{\underline{A}}(\underline{Q_i}, \underline{Q_j}) = \int_0^1 \underline{\underline{A}}(\underline{\psi}(s))\, ds$$

with segment path

$$\underline{\psi}(s) = \underline{Q_i} + s(\underline{Q_j} - \underline{Q_i})$$

We approximate this integral by the most basic integration, the midpoint rule.

$$\boxed{\underline{\underline{A}}(\underline{Q}_i, \underline{Q}_j) \approx \underline{\underline{A}}(\underline{\psi}(0.5)) = \underline{\underline{A}}\left(\tfrac{1}{2}(\underline{Q}_i + \underline{Q}_j)\right)}$$